Measure Theory 2014: Homework 1

Email your homework to Stein Bethuelsen,

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or, place it in Stein's mailbox.

Homework is due on 08.10.2014, 11.15am (sharp!!!)

Final grade: Homework (90%)+Attendance (10%).

(1) [20 points] Bauer, Chapter I, Ex. 2, p.13.

(2) [40 points] **Hausdorff measure and Hausdorff dimension on $\mathbb{R}^d$.** Let $C \subset \mathbb{R}^d$, $\epsilon > 0$ and $s \geq 0$.

$$\mathcal{H}^s_\epsilon(C) = \inf\Big\{\sum_{n=1}^\infty \Big(\mathrm{diam}(A_n)\Big)^s : \quad C \subseteq \bigcup_{n=1}^\infty A_n,\ \mathrm{diam}(A_n) < \epsilon\Big\},$$

where $\mathrm{diam}(S) = \sup_{x,y\in S} |x-y|$. Finally, let

$$(\star) \qquad \mathcal{H}^s(C) = \lim_{\epsilon\to 0} \mathcal{H}^s_\epsilon(C)$$

Prove that

- the limit $\epsilon \to 0$ in $(\star)$ exists;
- $\mathcal{H}^s(\cdot)$ is an outer measure:

$$\mathcal{H}^s(\emptyset) = 0,\ \mathcal{H}^s(C) \geq 0, \quad \mathcal{H}^s\left(\cup_n C_n\right) \leq \sum_n \mathcal{H}^s(C_n).$$

- For any $C, D \subset \mathbb{R}^d$ with $\mathrm{dist}(C,D) = \inf_{x\in C, y\in D} |x-y| > 0$,

$$\mathcal{H}^s(C \cup D) = \mathcal{H}^s(C) + \mathcal{H}^s(D).$$

- For any C, there exists $s_0 \in [0, +\infty]$ such that

$$\mathcal{H}^s(C) = \begin{cases} +\infty, & s < s_0, \\ 0, & s > s_0 \end{cases}$$

  This critical value s_0 is called the Hausdorff dimension of C, $s_0 = \dim_H(C)$.
- Compute the Hausdorff dimension of the middle-third Cantor set

$$C = [0,1] \setminus \bigcup_{n=1}^\infty \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right).$$

  In other words,

$$C = \lim_{n\to\infty} C_n,$$

  where

$$\begin{aligned} C_0 &= [0,1] \\ C_1 &= [0,1/3] \cup [2/3,1] \\ C_2 &= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1] \\ &\ldots \end{aligned}$$

  i.e., C_{n+1} is obtained by removing the "middle-thirds" from intervals forming C_n.

(3) [20 points] **Inner measures.** Suppose $\mathcal{A}$ is a σ-algebra and μ is a finite measure on $\mathcal{A}$. Define outer and inner measures as follows:

$$\mu^*(Q) = \inf\Big\{\sum_{n=1}^\infty \mu(A_n) : \quad \forall n,\ A_n \in \mathcal{A} \text{ and } Q \subseteq \bigcup_n A_n\Big\},$$

$$\mu_*(Q) = \sup\Big\{\mu(A) : \quad A \subset Q, A \in \mathcal{A}\Big\}.$$

Let $\bar{\mathcal{A}}$ be the collection of all sets Q, $Q \subseteq \Omega$, such that

$$\mu_*(Q) = \mu^*(Q).$$

Show that $\bar{\mathcal{A}}$ is a σ-algebra, and $\mu_*|_{\bar{\mathcal{A}}} = \mu^*|_{\bar{\mathcal{A}}}$ is a measure of $\bar{\mathcal{A}}$.

(4) [20 points] Bauer, Chapter I, Ex. 6a, p.26.