MARKOV CHAINS AND APPLICATIONS, MAIN EXAM

Exam 31/01/2024 (max 100 points)

(1) [20 points]
Let $(X_n)_{n\in\mathbb{N}_0}$ be an irreducible Markov chain on a finite state or countable space S.

(a) Show that either all states are recurrent or all the states are transient.
(b) Show that if a state $i \in S$ is transient, then $\sum_n p_{ii}^n \to 0$, as $n \to \infty$.

(2) [20 points]

(a) State the definition of irreducibility for discrete time Markov chains.
(b) Consider the Markov chain $(X_n)_{n\in\mathbb{N}_0}$, with state space $S = \{0, 1\}$, and with matrix of transition probabilities P given by

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Is this chain irreducible? Justify your answer.
Are the states recurrent or transient? Justify your answer.
(c) Let the initial distribution μ_0 be given by $\mu_0 = (3/7 \quad 4/7)$. For each $n \geq 1$, determine $\mathbb{P}(X_n = 0)$.

(3) [20 points]

(a) Let $(Y_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables such that $\mathbb{P}(Y = -1) = \mathbb{P}(Y = +1) = \frac{1}{2}$ where Y is an element of the sequence $(Y_n, n \in \mathbb{N})$. Define $S_0 := 0$ and $S_n := \sum_{i=1}^{n} Y_i$ for $n \in \mathbb{N}$. Decide whether the process $S = (S_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain with state space $\mathbb{Z}$ or not. Justify your answer.
(b) Provided that the answer to Q3(a) is affirmative, consider the state $0 \in \mathbb{Z}$. Is this state recurrent or transient? Justify your answer.

(4) [20 points] Let $X = (X_t)_{t\geq 0}$ be a homogeneous, continuous-time Markov chain with countable state space S. Suppose $X_0 = i$ for some $i \in S$. Let T_i denote the first time the process X moves away from state i. Show that T_i satisfies the memory-less property. That is, show that for all $s, t \geq 0$,

$$\mathbb{P}(T_i > s + t | T_i > s) = \mathbb{P}(T_i > t).$$

(5) [20 points] Suppose that each bacteria in a group of bacteria either splits into two new bacteria after an exponentially distributed time with parameter λ, or dies after an exponentially distributed time with parameter μ.

(a) Describe this as a birth-death process. What are the birth rates λ_i , and what are the death rates μ_i?
(b) Find a stationary distribution. Justify your answer.