MARKOV CHAINS AND APPLICATIONS, MAIN EXAM

Exam 11/12/2024 (max 100 points)

(1) [20 points]

(a) State the convergence theorem for finite state homogeneous discrete time Markov chains.

(b) Let $(X_n)_{n\in\mathbb{N}_0}$ be an aperiodic irreducible Markov chain on the finite state space $S = \{0, 1, \dots, M\}$, with a doubly stochastic transition matrix P. (Recall that a transition matrix is called doubly stochastic if the sum over the entries of each column is equal to 1.)

Show that the stationary distribution π is equal to the discrete uniform distribution on S. That is, show that

$$\pi_i = \frac{1}{M+1}, \text{ for each } i \in S.$$

(2) [20 points]

(a) State the definition of aperiodicity for discrete time Markov chains.

(b) Consider the Markov chain $(X_n)_{n\in\mathbb{N}_0}$, with state space $S = \{0, 1\}$, and with matrix of transition probabilities P given by

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

for $a, b \in (0, 1)$.

(i) Find P^n for $n \geq 1$.

(ii) Are the states recurrent or transient? Justify your answer.

(iii) Are the states aperiodic? Justify your answer.

(3) [20 points]

(a) Let N be some positive integer, and let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain on the state space $S = \{1, 2, \dots, N\}$, satisfying $p_{ii} = 1$ for each state $i \in S$. Show that any distribution on the state space S is a stationary distribution.

(b) Item (a) tells us that for a given Markov chain there can be more than one stationary distribution. Under which conditions does the chain have a unique distribution?

(4) [20 points] Let $X = (X_t)_{t\geq 0}$ be a homogeneous, continuous-time Markov chain with countable state space S. Let $(P(t))_{t\geq 0}$ be its semigroup. Show that $P(t)$ is continuous for all t, that is, show that for all $i, j \in S$ and for all $s, t \geq 0$,

$$|p_{ij}(t+s) - p_{ij}(t)| \leq 1 - p_{ii}(s).$$

(5) [20 points] Let $(Y_n)_{n\in\mathbb{N}_0}$ be a sequence of independent, identically distributed random variables with a discrete uniform distribution on $\{-1, 0, 1\}$. Set $X_n := Y_n + Y_{n+1}$ for $n \in \mathbb{N}_0$.
Decide if $X = (X_n)_{n\in\mathbb{N}_0}$ is a homogeneous discrete-time Markov chain or not. Justify your answer.