Fourier Analysis 2022/2023

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Exam 7 July 2023, from 13:15 to 16:15.
Contains 5 questions for 90 points.

Instructions

You can answer the questions in English or in Dutch. If your choice is Dutch, please feel free to use English terminology when convenient.

- If you use results from the book or from the homework sheets, formulate clearly what you are using and where it can be found.
- You can use the results of the earlier parts of a question, even if you have not solved these parts.
- Hints are provided for convenience, you can choose to use or not use them.
- This exam contains 5 questions for 90 points on 4 pages.
- The final grade equals as $1 + \text{total points}/10$.

We recommend you use the following convention for the Fourier transform on $\mathcal{S}(\mathbb{R})$:

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

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1. Let $f \in L^1(\mathbb{T})$. Show that the following statements hold.
   (a) (8 points) If f is even, then $\widehat{f}(k) = \widehat{f}(-k)$ for every $k \in \mathbb{Z}$. On the other hand, if f is odd, then $\widehat{f}(k) = -\widehat{f}(-k)$ for every $k \in \mathbb{Z}$.

Solution: Let f be even, we have

$$\begin{aligned}
\widehat{f}(-k) &= \int_{\mathbb{T}} f(x)e^{-i(-k)x}dx \\
&= \int_{x>0} f(x)e^{ikx}dx + \int_{x<0} f(x)e^{ikx}dx \\
&= \int_{x>0} f(-x)e^{ikx}dx + \int_{x<0} f(-x)e^{ikx}dx \\
&= \int_{x<0} f(x)e^{-ikx}dx + \int_{x>0} f(x)e^{-ikx}dx \\
&= \widehat{f}(k).
\end{aligned}$$

A similar argument for the case when f is odd.

(b) (10 points) If f attains only real values, then $\overline{\widehat{f}(k)} = \widehat{f}(-k)$ for every $k \in \mathbb{Z}$. Conversely, if f is continuous and $\overline{\widehat{f}(k)} = \widehat{f}(-k)$ for every $k \in \mathbb{Z}$, then f attains only real values.

> **Solution:** For the converse direction, one needs to look at the Fourier coefficients of $f - \overline{f}$ and use the uniqueness of Fourier coefficients.

(c) (15 points) Assume that f is α-Hölder continuous where $0 < \alpha \leq 1$: there is a constant $C > 0$ such that for every $x, y \in [-\pi, \pi]$ it holds

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}.$$

Then there is a constant $M > 0$ such that for every $k \in \mathbb{Z} \setminus \{0\}$ we have

$$|\widehat{f}(k)| \leq \frac{M}{|k|^{\alpha}}.$$

> **Solution:** Split the coefficients in sine and cosine and use the condition of being Hölder.

2. Consider the characteristic function $\chi_{[0,1]} : [-\pi, \pi] \to \mathbb{R}$ as a 2π-periodic function. Define

$$f := \chi_{[0,1]} * \chi_{[0,1]}.$$

(a) (4 points) Compute the partial sums of the Fourier series of $\chi_{[0,1]}$ and f.

> **Solution:** One first computes the convolution product $\chi_{[0,1]} * \chi_{[0,1]}$, which is equal to
>
> $$f(x) = \begin{cases} x, & x \in (0,1] \\ 2 - x, & x \in (1,2] \\ 0, & \text{otherwise} \end{cases}.$$
>
> Using the property of Fourier coefficients, we have $\widehat{\chi * \chi} = \widehat{\chi} \cdot \widehat{\chi}$, therefore we have
>
> $$\widehat{\chi}_{[0,1]}(k) = \int_0^1 e^{-ikx} dx = \frac{1 - e^{-ik}}{ik,}$$
>
> which implies that $\widehat{f}(k) = \frac{2e^{-ik} - e^{-2ik} - 1}{k^2}$, with which one can write the partial sum of the Fourier series.

(b) (8 points) Examine the point-wise convergence of the Fourier series of $\chi_{[0,1]} : [-\pi, \pi] \to \mathbb{R}$ and compute the limits when it makes sense.

> **Solution:** We use the theorem by Dini (Lecture 4). Fix $x \in [-\pi, \pi]$ and consider three cases:
>
> 1. $x = 0, 1$
> 2. $x \in (0, 1)$
> 3. $x \in [-\pi, \pi] \setminus [0, 1]$
>
> In the first case, we set $\beta = 1/2$. At $x = 0$, we have
>
> $$\int_0^{\pi} \left| \frac{\chi_{[0,1]}(0 + t) + \chi_{[0,1]}(0 - t)}{2} - \frac{1}{2} \right| \frac{d\lambda(t)}{t} = \int_1^{\pi} \frac{1}{2} \frac{d\lambda(t)}{t} = \frac{1}{2} \ln(\pi) < \infty.$$

It follows that the Fourier series converges at $x = 0$ and that the limit is equal to $1/2$. The case $x = 1$ can be handled analogously.
For the second case, we take $\beta = 1$. Since $(0, 1)$ is open, there is a $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq (0, 1)$. Hence

$$\begin{aligned}
\int_0^\pi \left| \frac{\chi_{[0,1]}(x+t) + \chi_{[0,1]}(x-t)}{2} - 1 \right| \frac{d\lambda(t)}{t} &= \int_\delta^\pi \left| \frac{\chi_{[0,1]}(x+t) + \chi_{[0,1]}(x-t)}{2} - 1 \right| \frac{d\lambda(t)}{t} \\
&\leq \int_\delta^\pi 2 \frac{d\lambda(t)}{t} \\
&< \infty.
\end{aligned}$$

Again, it follows that Fourier series converges at $x \in (0, 1)$ but now the limit is equal to 1.
The third case can be handled in the same way as case 2 with $\beta = 0$. We conclude that

$$\lim_N S_N^f(x) = \begin{cases} 1, & x \in (0,1) \\ 1/2, & x = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

3. (15 points) Let $f \in L^1(\mathbb{T})$. Show that the operator $A_f : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ defined by $A_f(g) := f * g$ satisfies

$$\|A_f\| := \sup \{\|A_f(g)\|_1 \ : \ \|g\|_1 \leq 1\} = \|f\|_1.$$

Hint: Use the Fejér kernel.

Solution: Let $g_N = F_N$ be the Fejér kernel in the lecture note. Then we have $A_f(g_N) = A_f(F_N) = \sigma_N * f$ and $\|\sigma_N(f) - f\|_1 \to 0$ uniformly therefore we have $\|A_f\| \geq \|f\|_1$. On the other hand, we have

$$\begin{aligned}
\|A_f(g)\|_1 &= \|f * g\|_1 \\
&= \int_\mathbb{T} | \int_\mathbb{T} f(y) g(x-y) dy | dx \\
&\leq \int_\mathbb{T} \int_\mathbb{T} |f(y)| \cdot |g(x-y)| dy dx \\
&= \int_\mathbb{T} \int_\mathbb{T} |f(y)| \cdot |g(x-y)| dx dy \\
&= \|f\|_1 \cdot \|g\|_1 \implies \|A_f\| \leq \|f\|_1,
\end{aligned}$$

where the second last equality is due to Fubini.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = xe^{-x^2}$.

   (a) (6 points) Show that the Fourier transform is $\widehat{f}(\xi) = -i\frac{\sqrt{\pi}}{2}\xi e^{-\frac{\xi^2}{4}}$.

   Solution: Observe that the function xe^{-x^2} is the derivative of $-\frac{1}{2}e^{-x^2}$. Compute the Fourier transform of e^{-x^2} and use $\widehat{g'}(\xi) = i\xi\widehat{g}(\xi)$.

   (b) (6 points) Calculate

$$\int_\mathbb{R} |(f * f)(x)|^2 dx.$$

Solution: Use the Plancerel formula.

Hint: You may use that $\int_{\mathbb{R}} e^{-t^2} dt = \sqrt{\pi}$ and that $\int_{\mathbb{R}} t^4 e^{-t^2} dt = \frac{3\sqrt{\pi}}{4}$.

5. Recall that for $u \in \mathcal{S}'(\mathbb{R})$, the support of u, denoted $\operatorname{supp}(u)$ is

$$\operatorname{supp}(u) := \mathbb{R} \setminus \bigcup \{I : u[\varphi] = 0 \ \forall \varphi \in \mathcal{S}(\mathbb{R}) : \operatorname{supp}(\varphi) \subseteq I\}.$$

(a) (8 points) Let $u \in \mathcal{S}'(\mathbb{R})$ and assume that $\operatorname{supp}(\widehat{u})$ is compact. Find a function $f \in \mathcal{S}(\mathbb{R})$ such that $u * f = u$.

Solution: Take $g \in \mathcal{S}(\mathbb{R})$ such that $g(x) = 1$ for all $x \in \operatorname{supp}(\widehat{u})$ and let $f = \widetilde{\widehat{g}}$. Then

$$\widehat{u * f}[\psi] = u * f[\widehat{\psi}] = u[\widetilde{f} * \widehat{\psi}] = u[\widehat{g} * \widehat{\psi}] = u[\widehat{g\psi}] = \widehat{u}[g\psi] = \widehat{u}[\psi].$$

Since the Fourier transform is an isomorphism, it follows that $u * f = u$.

(b) (10 points) Use the preceding part to show that if $u \in L^2(\mathbb{R})$ and $|\xi| \leq \lambda$ for $\xi \in \operatorname{supp}(\widehat{u})$, then

$$\|u'\|_2 \leq C \cdot \lambda \cdot \|u\|_2,$$

where C is a constant.

Solution: First note that

$$\widehat{f}(\xi) = \int \widetilde{\widehat{g}}(x) e^{-i\xi x} dx = \int \widehat{g}(-x) e^{-i\xi x} dx = \int \widehat{g}(x) e^{i\xi x} dx = \widetilde{\widehat{\widehat{g}}}(\xi) = 2\pi g(\xi).$$

Hence, $\widehat{f}(\xi) = 2\pi$ on $\operatorname{supp}(\widehat{u})$. Now, using (a) and the Plancherel formula, we obtain

$$\begin{aligned}
\|u'\|_2^2 &= \|u * f'\|_2^2 \\
&= \int |u * f'(x)|^2 dx \\
&= \frac{1}{2\pi} \int |\widehat{u * f'}(\xi)|^2 d\xi \\
&= \frac{1}{2\pi} \int |\widehat{u} \widehat{f'}(\xi)|^2 d\xi \\
&= \frac{1}{2\pi} \int \xi^2 |\widehat{u}(\xi) \widehat{f}(\xi)|^2 d\xi \\
&\leq \frac{(2\pi\lambda)^2}{2\pi} \int |\widehat{u}(\xi)|^2 d\xi \\
&= (2\pi\lambda)^2 \int |u(x)|^2 dx.
\end{aligned}$$

So $\|u'\|_2 \leq 2\pi \cdot \lambda \cdot \|u\|_2$.