Fourier Analysis 2022/2023

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Exam 16 June 2023, from 13:15 to 16:15.
Contains 5 questions for 99 points.

Instructions

You can answer the questions in English or in Dutch. If your choice is Dutch, please feel free to use English terminology when convenient.

- If you use results from the book or from the homework sheets, formulate clearly what you are using and where it can be found.
- You can use the results of the earlier parts of a question, even if you have not solved these parts.
- Hints are provided for convenience, you can choose to use or not use them.
- This exam contains 5 questions for 99 points on 8 pages. The final grade equals as $1 +$ total points/11.

We recommend you use the following convention for the Fourier transform on $\mathcal{S}(\mathbb{R})$:

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

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1. (a) (5 points) Denote by $C^1(\mathbb{T}, \mathbb{R})$ the space of real-valued 2π periodic C^1-functions. Suppose that $f \in C^1(\mathbb{T}, \mathbb{R})$ satisfies $\int_{\mathbb{T}} f(t)d\lambda(t) = 0$. Show that

$$\int_{\mathbb{T}} |f(t)|^2 d\lambda(t) \leq \int_{\mathbb{T}} |f'(t)|^2 d\lambda(t).$$

   (b) (10 points) Prove that equality holds if and only if there are constants $c_1, c_2 \in \mathbb{R}$ such that $f(t) = c_1 \sin(t) + c_2 \cos(t)$.

   Hint: Use Parseval's identity.

Solution:

Noting that $\widehat{f'}(n) = in\widehat{f}(n)$ and applying Parseval's identity to f and f' yields

$$\frac{1}{2\pi} \int |f(x)|^2 dx = \sum_{n \neq 0} |\widehat{f}(n)|^2, \quad \frac{1}{2\pi} \int |f'(x)|^2 dx = \sum_{n \neq 0} n^2 |\widehat{f}(n)|^2.$$

Hence

$$\int |f(x)|^2 dx = 2\pi \sum_{n\neq 0} |\widehat{f}(n)|^2 \leq 2\pi \sum_{n\neq 0} n^2 |\widehat{f}(n)|^2 = \int |f'(x)|^2 dx.$$

Now, assume $\int |f(x)|^2 dx = \int |f'(x)|^2 dx$. Again, we note that $\widehat{f'}(n) = in\widehat{f}(n)$ and using Parseval's identity, we conclude that

$$\sum_{n\neq 0} |\widehat{f}(n)|^2 = \frac{1}{2\pi} \int |f(x)|^2 dx = \frac{1}{2\pi} \int |f'(x)|^2 dx = \sum_{n\neq 0} n^2 |\widehat{f}(n)|^2. \tag{1}$$

Towards a contradiction, suppose that $\widehat{f}(k) \neq 0$ for some $|k| > 1$, then

$$k^2 > 1 \implies k^2 |\widehat{f}(k)|^2 > |\widehat{f}(k)|^2.$$

It follows that

$$\sum_{n\neq 0,k} |\widehat{f}(n)|^2 + |\widehat{f}(k)|^2 < \sum_{n\neq 0,k} n^2 |\widehat{f}(n)|^2 + k^2 |\widehat{f}(k)|^2,$$

which contradicts equation (1). So the only Fourier coefficients of f that may be nonzero are $\widehat{f}(1)$ and $\widehat{f}(-1)$. By uniqueness of Fourier series (Lecture 4), f is given by $f(x) = \widehat{f}(1)e^{ix} + \widehat{f}(-1)e^{-ix}$. Rewriting f in the basis $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{N}}$, we have $f(x) = c_1 \cos(x) + c_2 \sin(x)$, where

$$c_1 = \frac{1}{\pi} \int f(x) \cos(x) dx, \quad c_2 = \frac{1}{\pi} \int f(x) \sin(x) dx$$

are clearly real numbers. Alternatively, you may note that $\widehat{f}(-1) = \overline{\widehat{f}(1)}$ and use Euler's identity to write $\widehat{f}(1)e^{ix} + \widehat{f}(-1)e^{-ix}$ in the desired form.

2. Let $f \in L^1(\mathbb{T})$. Show the following.

   (a) (5 points) For every $x \in \mathbb{T}$ and $N \in \mathbb{N}$ the Cesàro mean $\sigma_N(f)$ can be written as

$$\sigma_N(f)(x) = \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \widehat{f}(k) e^{ikx}.$$

Solution: This is the rewriting of the Cesàro mean: Firstly we have

$$s_N(f)(x) = \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx}.$$

Then the Cesàro mean is the following:

$$\begin{aligned}
\sigma_N &= \frac{1}{N}\sum_{k=0}^{N-1} s_k(f) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=-k}^{k} \widehat{f}(n)e^{inx} \\
&= \frac{1}{N}(\widehat{f}(0)e^{i\cdot 0\cdot x} + \widehat{f}(-1)e^{i\cdot(-1)\cdot x} + \widehat{f}(0)e^{i\cdot 0\cdot x} + \widehat{f}(1)e^{i\cdot 1\cdot x} + \cdots + \widehat{f}(N-1)e^{i\cdot(N-1)\cdot x} \\
&= \frac{1}{N}(N\cdot\widehat{f}(0)e^{i\cdot 0\cdot x} + (N-1)\widehat{f}(-1)e^{-ix}) + (N-1)\widehat{f}(1)e^{ix} + \cdots + \widehat{f}(N-1)e^{i(N-1)x} \\
&= \sum_{k=-(N-1)}^{N-1} (\frac{N-|k|}{N})\widehat{f}(k)e^{ikx} \\
&= \sum_{k=-(N-1)}^{N-1} (1-\frac{|k|}{N})\widehat{f}(k)e^{ikx}.
\end{aligned}$$

(b) (15 points) If $\int_{\mathbb{T}} f(t)d\lambda(t) = 0$, define the 2π-periodic function $g : [-\pi,\pi] \to \mathbb{C}$ as

$$g(x) = \int_{-\pi}^{x} f(t)d\lambda(t).$$

It is 2π-periodic since $g(-\pi) = g(\pi)$ (this does not need to be proved). Prove that

1. g is continuous and hence a function in $C(\mathbb{T})$;

Solution: Since $C_c(\mathbb{T})$ is dense in $L^1(\mathbb{T})$ (with respect to L^1-norm) and by the compactness of $\mathbb{T}$ we have $C_c(\mathbb{T}) = C(\mathbb{T})$. Then for any $\epsilon > 0$, find $h \in C(\mathbb{T})$ such that $\|h - f\|_1 \leq \epsilon$. We have the following estimate:

$$\begin{aligned}
|g(x+\Delta x) - g(x)| &= |\int_{x}^{x+\Delta x} f(t)d\lambda(t)| \\
&= |\int_{x}^{x+\Delta x} f(t) - h(t) + h(t)d\lambda(t)| \\
&\leq |\int_{x}^{x+\Delta x} h(t)d\lambda(t)| + \|f-h\|_1 \\
&\leq \sup_{x\in\mathbb{T}} |h(t)|\cdot\Delta x + \epsilon \to \epsilon
\end{aligned}$$

Since this holds for arbitrarily small ϵ, we prove the continuity of g.

2. $\widehat{f}(k) = ik\widehat{g}(k)$ for every $k \in \mathbb{Z}$.

Solution:
Let $\{f_n\}$ be the sequence of continuous functions on $\mathbb{T}$ that converges to f in L^1 norm (therefore $\widehat{f_n}(k)$ converges to $\widehat{f}(k)$ uniformly for each $k \in \mathbb{Z}$). By compactness of $\mathbb{T}$ (or the fact that $\mathbb{T}$ has finite measure), define $g_n(x) := \int_{-\pi}^{x} f_n(t)d\lambda(t)$, we have that $g_n(x)$

converges to $g(x)$ in L^1 norm as well. Indeed, we have

$$\begin{aligned} \|g - g_n\|_1 &= \int_{\mathbb{T}} |\int_{-\pi}^{x} f_n(t) - f(t) d\lambda(t)| d\lambda(x) \\ &\leq \int_{\mathbb{T}} |\int_{-\pi}^{x} |f_n(t) - f(t)| d\lambda(t)| d\lambda(x) \\ &= 2\pi \|f_n - f\|_1 \to 0. \end{aligned}$$

Since f_n is continuous, we have g_n is continuously differentiable and $g_n' = f_n$. By the property of Fourier coefficient of derivative, we have $\widehat{f}_n(k) = ik\widehat{g}_n(k)$. Since L^1 convergence implies the uniform convergence of the Fourier coefficient, we have

$$\widehat{f}(k) = \lim_n \widehat{f}_n(k) = \lim_n ik\widehat{g}_n(k) = ik\widehat{g}(k).$$

(c) (10 points) If $\int_{\mathbb{T}} f(t) d\lambda(t) = 0$ and $\widehat{f}(k) = -\widehat{f}(-k) \geq 0$ for every $k \in \mathbb{Z}$, then it holds that

$$\sum_{k=1}^{\infty} \frac{1}{k} \widehat{f}(k) < \infty.$$

Hint: Use parts (a) and (b) together with Fejér's Theorem on the function g defined from f as in part (b).

Solution: Since g is continuous, we can apply Fejér's theorem for g:

$$\lim_N \sigma_N(g)(x) = g(x) < \infty.$$

To prove the desired inequality, we estimate the following:

$$\begin{aligned} \sigma_N(g)(0) &= \sum_{k=-(N-1)}^{N-1} (1 - \frac{|k|}{N}) \widehat{g}(k) e^{i \cdot k \cdot 0} \\ &= \widehat{g}(0) + \sum_{k=-(N-1)}^{N-1, k \neq 0} (1 - \frac{|k|}{N}) \frac{1}{ik} \widehat{f}(k) \\ &= \widehat{g}(0) + 2 \sum_{k=1}^{N-1} (1 - \frac{|k|}{N}) \frac{1}{ik} \widehat{f}(k) \\ &= \widehat{g}(0) + 2 \sum_{k=1}^{N-1} \frac{1}{ik} \widehat{f}(k) - \frac{2}{iN} \sum_{k=1}^{N-1} \widehat{f}(k). \end{aligned}$$

Since $\widehat{f}(k) \to 0$ we have $\forall \epsilon > 0$, there exists $N_\epsilon > 0$ such that $|\widehat{f}(k)| \leq \epsilon$ for all $k \geq N_\epsilon$. Let $M = \max_{k \leq N_\epsilon} |\widehat{f}(k)|$, we then have

$$g(0) = \lim_N \frac{1}{N} \sum_{k=1}^{N} \widehat{f}(k) \leq \lim_N \frac{N_\epsilon \cdot M}{N} + \frac{\epsilon \cdot (N - N_\epsilon)}{N} = \epsilon,$$

which holds for arbitrarily small ϵ. Therefore, we have

$$\lim_N \sigma_N(g)(0) = \lim_N (\widehat{g}(0) + 2 \sum_{k=1}^{N-1} \frac{1}{ik} \widehat{f}(k) + \frac{2}{iN} \sum_{k=1}^{N-1} \widehat{f}(k)) = \widehat{g}(0) + 2 \sum_{k=1}^{\infty} \frac{1}{ik} \widehat{f}(k),$$

which implies that

$$|\sum_{k=1}^{\infty}\frac{1}{ik}\widehat{f}(k)| = \sum_{k=1}^{\infty}\frac{1}{k}\widehat{f}(k) = |\frac{g(0)-\widehat{g}(0)}{2}| < \infty.$$

3. Consider the tempered distribution induced by the function

$$f(x) = |x^2-2|.$$

(a) (4 points) Compute its Fourier transform;

Solution: Observe that the function $f(x) = |x^2-2|$ can be rewritten as the following:

$$(x^2-2)(1+2H(x-\sqrt{2})-2H(x+\sqrt{2}))),$$

where the function $H(x)$ is the Heaviside function introduced either in the lecture note or in the assignment.
Since the Fourier transform of the distribution is given by $\widehat{F}(f) := F(\widehat{f})$, we have

$$\begin{aligned}\widehat{F}_f(g) &= \int_{\mathbb{R}} |x^2-2|\widehat{g}dx \\ &= \int_{\mathbb{R}} (x^2-2)(1+2H(x-\sqrt{2})-2H(x+\sqrt{2})))\widehat{g}dx \\ &= \int_{\mathbb{R}} (x^2-2)\widehat{g}dx + \int_{\mathbb{R}} (2H(x-\sqrt{2})-2H(x+\sqrt{2}))(x^2-2)\widehat{g}dx \\ &= \int_{\mathbb{R}} -\widehat{g''}-2\widehat{g}dx + \int_{\mathbb{R}} (2H(x-\sqrt{2})-2H(x+\sqrt{2}))(-\widehat{g''}-2\widehat{g})dx \\ &= \int_{\mathbb{R}} -\widehat{g''}-2\widehat{g}dx + F_{2H(x-\sqrt{2})-2H(x+\sqrt{2})}(\widehat{-g''-2g}).\end{aligned}$$

Since $2H(x-\sqrt{2})-2H(x+\sqrt{2})$ is L^1 function, we have for L^1 functions f, $\widehat{F}_f(g) = F_{\widehat{f}}(g)$.
Therefore, we have by using the Fourier inversion formula:

$$\widehat{F}_f(g) = -2\pi(g''(0)+2g(0)) + F_{\widehat{2H(x-\sqrt{2}-2H(\sqrt{x+\sqrt{2}}))}}(g).$$

(b) (5 points) Compute its second derivative in the sense of distributions.

Solution: This exercise is to check if students understand the Leibniz rule for distributions: $(XF)' = X'F + XF'$ when X is a multiplicator.
Note that $(F_H)' = \delta$ where H is the Heaviside function.

$$F' = 2x(1+2H(x-\sqrt{2})-2H(x+\sqrt{2})) + (x^2-2)(2\delta(x-\sqrt{2})-2\delta(x+\sqrt{2})).$$

Therefore we have

$$\begin{aligned}F'' =& 2(1+2H(x-\sqrt{2})-2H(x+\sqrt{2})) \\ &+4x(2\delta(x-\sqrt{2})-2\delta(x+\sqrt{2})) \\ &+2(x^2-2)(2\delta(x-\sqrt{2})-2\delta(x+\sqrt{2}))\circ\frac{d}{dx}.\end{aligned}$$

4. Consider the operator $\Phi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ given by $\Phi f = \frac{1}{\sqrt{2\pi}}\widehat{f}$.

(a) (5 points) Show that $\Phi^4 = 1$, the identity operator on $\mathcal{S}(\mathbb{R})$;

Solution: Take $\psi \in \mathcal{S}(\mathbb{R})$. Recall that $\widehat{\widehat{\psi}} = 2\pi\widetilde{\psi}$, where $\widetilde{\psi}(x) = \psi(-x)$. Therefore,

$$\Phi^2(\psi) = \frac{1}{2\pi}\widehat{\widehat{\psi}} = \frac{2\pi}{2\pi}\widetilde{\psi} = \widetilde{\psi}.$$

It follows that

$$\Phi^4(\psi) = \Phi^2(\Phi^2(\psi)) = \Phi^2(\widetilde{\psi}) = \widetilde{\widetilde{\psi}} = \psi,$$

i.e. $\Phi^4(\psi) = \psi$.

(b) (10 points) Show that every $f \in \mathcal{S}(\mathbb{R})$ has a unique decomposition as

$$f = \sum_{k=0}^{3} f_k, \qquad f_k \in \mathcal{S}(\mathbb{R}),\ \Phi(f_k) = i^k f_k.$$

Solution: Fix $f \in \mathcal{S}(\mathbb{R})$ and let

$$f_k = \frac{1}{4}\sum_{n=0}^{3} i^{-nk}\Phi^n(f).$$

We have

$$\begin{aligned}\sum_{k=0}^{3} f_k =& \frac{1}{4}[f(1+1+1+1)+\\ &\Phi(f)(1+i^{-1}+i^{-2}+i^{-3})+\\ &\Phi^2(f)(1+i^{-2}+1+i^{-2})+\\ &\Phi^3(f)(1+i^{-3}+i^{-2}+i^{-1})]\\ =& f+0+0+0\\ =& f\end{aligned}$$

and

$$\begin{aligned}\Phi(f_k) &= \Phi\left(\frac{1}{4}\sum_{n=0}^{3} i^{-nk}\Phi^n(f)\right)\\ &= \frac{1}{4}\sum_{n=0}^{3} i^{-nk}\Phi^{n+1}(f)\\ &= \frac{i^k}{4}\sum_{n=0}^{3} i^{-(n+1)k}\Phi^{n+1}(f)\\ &= \frac{i^k}{4}\sum_{m=1}^{4} i^{-mk}\Phi^{m}(f)\\ &= \frac{i^k}{4}\sum_{m=0}^{3} i^{-mk}\Phi^{m}(f)\\ &= i^k f_k.\end{aligned}$$

(c) (5 points) Show that the differential operator $L(f)(x) = xf(x) + f'(x)$ satisfies

$$L\mathcal{F}(f) = i\mathcal{F}(L(f)),$$

and prove that $\Phi L(f_k) = i^{k+1}L(f_k)$ for the f_k's in the decomposition.

Solution: Note that for any $f \in \mathcal{S}(\mathbb{R})$, we have

$$\Phi(L(f)) = \Phi(xf(x) + f'(x)) = i\Phi(f)' + i\xi\Phi(f) = iL(\Phi(f)),$$

by using properties of the Fourier transform. Now, using the property of f_k from (b), it follows that

$$\Phi(L(f_k)) = iL(\Phi(f_k)) = iL(i^k f_k) = i^{k+1}L(f_k).$$

5. The following statements manifest that a non-zero function and its Fourier transform cannot be localised simultaneously to arbitrary precision.

(a) (10 points) Prove that if a non-zero $f \in C_0(\mathbb{R})$ has compact support, then $\widehat{f} \in C_0(\mathbb{R})$ **cannot** have also compact support. **Hint:** If you assume that f is compactly supported, you may extend it to a periodic function.

Solution: Towards a contradiction, suppose that f and $\widehat{f}$ are both compactly supported. Say $\operatorname{supp} f \subseteq [-A, A]$. Viewing f as a function on $[-2A, 2A]$, we may extend it to a $4A$-periodic function $\widetilde{f}$. Note that the Fourier coefficients c_n of $\widetilde{f}$ are given by

$$c_n = \frac{1}{4A}\int_{-2A}^{2A} \widetilde{f}(x)e^{-2\pi inx/4A}dx = \frac{1}{4A}\int_{-\infty}^{\infty} f(x)e^{-2\pi inx/4A}dx = \frac{1}{4A}\widehat{f}(n/4A).$$

Since $\widehat{f}$ is compactly supported $\exists N > 0$ such that $\forall n$ with $|n| > N$, we have $\widehat{f}(n/4A) = 0$. Since f, and so also $\widetilde{f}$, is continuous, uniqueness of Fourier coefficients implies that $\widetilde{f}(x) = \sum_{n=-N}^{N} c_n e^{-inx}$ is a trigonometric polynomial. However, $\widetilde{f}|_{[A,2A]} = 0$, which cannot happen for trigonometric polynomials. Thus, we have reached a contradiction and we conclude that both f and $\widehat{f}$ cannot be compactly supported.

(b) (15 points) Prove *Heisenberg's Uncertainty Principle:* Let $f \in \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ and assume for simplicity that it attains only real values and that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1.$$

Then, it holds that

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{f}(\xi)|^2 d\xi\right) \geq \frac{\pi}{2}.$$

Hint: Do integration by parts on $\int_{-\infty}^{\infty} |f(x)|^2 dx$ and then apply the Cauchy–Schwarz inequality for the inner product

$$\langle f, g\rangle_2 = \int_{\mathbb{R}} f(x)g(x)dx.$$

Solution: We compute

$$1 = \int f^2 = \int f^2 x' = [\text{Int. by parts}] = [f^2(x)x]_{-\infty}^{\infty} - 2\int x f f' = 0 - 2\langle xf, f' \rangle.$$

Using Cauchy-Scwartz, we find

$$\frac{1}{4} = |\langle xf, f' \rangle|^2 \leq \|xf\|^2 \|f'\|^2 = \int (xf)^2 \int (f')^2$$

Applying the Plancharel formula to the right factor in the last expression yields

$$\frac{1}{4} \leq \int (xf)^2 \frac{1}{2\pi} \int (\xi \widehat{f})^2 \implies \frac{\pi}{2} \leq \int (xf)^2 \int (\xi \widehat{f})^2$$

References

[1] A. Vretblad, *Fourier analysis and its applications*, Graduate Texts in Mathematics, 223, Springer-Verlag, New York, 2003. MR1992764