Exam Introduction to Perturbation Methods

20 June 2023, 13:15-16:15

- Please put your name and student number on every page you hand in.
- Do not just give answers, explain every step that you make.
- **NOTE that when a first-term expansion is requested, you need to determine the first non-zero term**.
- This exam consists of **three** assignments.

Good luck!

1. Consider the following problem

$$\varepsilon y'' + (1+2x)y' - 2y = 1, \tag{1}$$

for $0 < x < 1$, where $y(0) = \varepsilon$ and $y(1) = \sin(\varepsilon)$.

(a) Determine a regular expansion of this equation. Show that this solution cannot satisfy both boundary conditions.

(b) Explain where a boundary or interior layer lies.

(c) Determine a first-term expansion of the solution in this layer.

(d) Match the solutions that you found in (a) and (c) to each other. Give the first-term composite expansion of the problem.

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2. Consider the problem of solving

$$\dot{y} = y - y^3$$

for $t > 0$ where $y(0) = \varepsilon$.

(a) Sketch the direction field for this problem and from this determine what happens to the solution as $t \to \infty$.

(b) Suppose one assumes a regular expansion of the form $y(t) = \varepsilon y_0(t) + \varepsilon^\alpha y_1(t) + \cdots$ with $\alpha > 1$. After finding y_0 and y_1, explain why this is not expected to be an accurate approximation as $t \to \infty$. Also, explain why a multiplescale expansion should be used but the standard approach taking $t_1 = t$ and $t_2 = \varepsilon^\beta t$ will not work for this problem.

(c) Assume $t_2 = \varepsilon^\beta f(t)$. Determine $f(t)$.
Hint. $f(t)$ can be determined from the secular-removing condition and the requirement that $f(0) = 0$.

(d) Show that

$$y = \frac{\varepsilon e^t}{\sqrt{1 + \varepsilon^2(e^{2t} - 1)}},$$

to leading order.

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3. Consider for $\mu \in \mathbb{R}$, $h(x,\mu;\varepsilon) : \mathbb{R}^2 \to \mathbb{R}$ sufficiently smooth, and $0 < \varepsilon \ll 1$,

$$\dot{x} = x^4 - 5\mu x^2 + 4\mu^2 + \varepsilon h(x,\mu;\varepsilon) \tag{2}$$

(a) **The unperturbed case** $h(x,\mu;\varepsilon) \equiv 0$**.**
(i) Show that 4 critical points $\bar{x} = \pm\bar{x}_j(\mu)$, $(j = 1,2)$ appear simultaneously as μ increases through 0 and determine the stability of these critical points. Give a sketch of the associated bifurcation diagram, i.e. sketch $\pm\bar{x}_j(\mu)$ as functions of μ $(j = 1,2)$ and indicate the stability of the various branches.
(ii) In a standard, generic, setting 'new' critical points are created in pairs (this is the classical saddle-node bifurcation). This implies that a certain non-degeneracy condition must be violated in the present setting. Which one?

(b) **Perturbation analysis for general** $h(x,\mu;\varepsilon)$**.**
To investigate the impact of the 'unfolding' term $\varepsilon h(x,\mu;\varepsilon)$ on the bifurcational structure found in (a), we consider the effect of this perturbation on the 4 leading order branches $\bar{x} = \pm\bar{x}_j(\mu)$, $(j = 1,2)$ found in (a). Using the fact that x only appears in even powers in the leading order ($= \mathcal{O}(1)$) part of the equation for $\dot{x}$, we introduce $X_j = (\bar{x}_j)^2$ and expand: $X_j^\varepsilon(\mu) = X_j^0 + \varepsilon X_j^1 + \mathcal{O}(\varepsilon^2)$ with $X_j^0 = X_j^0(\mu) = (\bar{x}_j^0(\mu))^2$ and $\bar{x}_j^0(\mu)$ as found in (a). Show that $X_j^1 = X_j^1(\mu)$ is determined by $(2X_j^0 - 5\mu)X_j^1 = -h(\pm\sqrt{X_j^0}, \mu; 0)$.

(c) **The cases** $h(x,\mu;\varepsilon) = H_\pm(x,\mu;\varepsilon) = \pm 3\mu - \varepsilon$**.**
(i) $h = H_-$**.** Show that this perturbation indeed 'unfolds' the degenerate situation of (a) into a combination of 2 standard saddle-node bifurcations at $\mu = \mu_k^{SN}$, $k = 1,2$. Determine μ_k^{SN}, $k = 1,2$ (at leading order in ε), determine the stability of the critical points, and give a sketch of the bifurcation diagram (including stability properties).
Hint. Use (b).
(ii) $h = H_+$**.** As (c)(i), including the sketch of the bifurcation diagram (indicating the obtained stability properties). Show that there is a third bifurcation value μ_* in this case and determine a leading order approximation of μ_*.

(d) **Higher order effects?**
The 'sensitivity' of the unperturbed system is also shown by the fact that it is crucial to include the $-\varepsilon$ terms in $H_\pm(x,\mu;\varepsilon)$ – an $\mathcal{O}(\varepsilon^2)$ effect in (2) we sofar did not take into account. This can (for instance) be seen by dropping the $-\varepsilon$ term in $H_+(x,\mu;\varepsilon)$: take $h(x,\mu;\varepsilon) = 3\mu$ and determine a *local* bifurcation diagram, i.e. a bifurcation diagram (including stability properties) in which x and μ are asymptotically small.
Hint. Determine appropriate scalings for x and μ (for instance based on the outcome of the analysis in (c)).