Algebraic Curves, final exam

January 9th, 2023

- No calculators, cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated.
- You have 180 minutes.
- There are 4 questions for 38 points in total; your grade will be $\frac{\#\text{points received}}{38} \cdot 9 + 1$.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- If you cannot solve a part of an exercise which is needed later, you may write e.g. "Let N be the answer to part (a). Then ..."
- Good luck!

Throughout, we work over an algebraically closed field k. We assume the characteristic of k is not 2 or 3.

Exercise 1. Consider the affine plane curves given by

$$F = y - x^2$$
$$G = y - x^3$$

(a) (1 point) Find the intersection points of F and G.

(b) (6 points) Compute the intersection multiplicity at each intersection point. If you want to say that two curves have different tangent directions at a point, you don't need to prove this.

(c) (4 points) Consider the projective closures F^*, G^* of the curves above given by homogenizing the equations. Compute the intersection multiplicity at each intersection point of F^* and G^* in $\mathbb{P}^2$.

Exercise 2. (12 points) In each case, determine if the variety X is affine or projective. If X is affine, find the coordinate ring $\Gamma(X) = \Gamma(X, \mathcal{O}_X)$. If X is projective, find the homogeneous coordinate ring $\Gamma_h(X)$. (None of the examples are both affine and projective.)

In each case, find the ring of rational functions $k(X)$ and write down the dimension of each variety.

(a) $X = \mathbb{A}^1$

(b) $X = \mathbb{P}^1$

(c) $X = \mathbb{P}^3$

(d) $X = V(F)$ inside $\mathbb{P}^2$, where F is the form $x^2 + y^2 + z^2$.

Exercise 3. Let $E = V(F) \subseteq \mathbb{P}^2$ be the vanishing locus of the cubic equation

$$zy^2 = x^2(x + z).$$

Assume the characteristic of k is not 2 or 3.

(a) (2 points) What is the multiplicity of E at the origin $[0:0:1]$?

(b) (3 points) Give the equations for the tangent lines to E at $[0:0:1]$. Is the origin an ordinary double point of the curve E?

(c) (4 points) Let $E' := E \setminus \{[0:0:1]\}$. All points of E' are simple and E' has a group law with identity $[0:1:0]$, defined similarly to that of a nonsingular cubic curve. (The group law on the simple points E' of a cubic curve was defined in your homework.)
What is the sum of the point $p = [-1:0:1]$ with itself under this group law?

Exercise 4. (6 points) Let X be the subvariety of $\mathbb{P}^2$ given by $zx - y^2 = 0$. Consider the maps $f, g\colon X \dashrightarrow \mathbb{P}^1$ given by

$$f([x:y:z]) = [x:y]$$
$$g([x:y:z]) = [y:z].$$

Show that f and g represent the same rational map from X to $\mathbb{P}^1$. What is the domain of this map?