Linear Analysis

27 January 2023 13:15-16:15hr

- You can use the results of the earlier parts of a question, even if you have not solved these parts.
- Write in full sentences and clearly state what you use in your solutions.
- The point distribution is preliminary and may be subject to change.
- This exam has five questions on two pages.

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1. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis in H. Let $T \in B(H)$ be a bounded operator. For $k \in \mathbb{N}$ define $T_k : H \to H$ by

$$T_k(x) := \sum_{n=0}^{k} \langle x, e_n \rangle T e_n.$$

   a. Prove that T_k is a bounded linear operator.

   b. Prove that for all $x \in H$ the sequence $T_k x$ converges to Tx in norm on H.

   c. Prove that $\sup_k \|T_k\| < \infty$.

   d. Suppose that $T_k \to T$ as $k \to \infty$ in the operator norm on $B(H)$. Prove that $Te_n \to 0$ as $n \to \infty$ in norm on H.

2. Let X be a Banach space and $\{X_n\}_{n=0}^{\infty}$ a countable collection of linear subspaces of X such that $X = \bigcup_{n=0}^{\infty} X_n$.

   a. Prove that there exist $k \geq 0$ and $\varepsilon > 0$ such that $B(0, \varepsilon) \subset \overline{X}_k$.

   b. If k is as in part a.), prove that $X = \overline{X}_k$.

   Let H be an inner product space and $\{e_n\}_{n=0}^{\infty}$ an orthonormal sequence such that

   - for all $x \in H$ there exists $N \geq 0$ such that $\langle x, e_n \rangle = 0$ for all $n \geq N$;
   - $\{e_n : n \in \mathbb{N}\}^{\perp} = \{0\}$.

   c. Prove that H is not a Hilbert space.

3. Let X be a Banach space and $S \in B(X)$ a bijective linear transformation.

   a. Prove that $\|x\|_S := \|Sx\|$ is a norm on X.

   b. Prove that the norms $\|\cdot\|$ and $\|\cdot\|_S$ are equivalent.

4. Let X be a Banach space, $A \subset X$ a subset and $T \in B(X)$ a bounded operator. We write

$$TA := \{Ta \in X : a \in A\} \subset X,$$

   for the *image of A under T*. Furthermore we write $\overline{TA}$ for the closure of the set TA. A subset $B \subset X$ is *bounded* if

$$\sup\{\|b\| : b \in B\} < \infty.$$

   We say that the operator T is *compact* if it has the following property: For every bounded subset $B \subset X$ the set $\overline{TB}$ is compact.

a. Prove that X is finite dimensional if and only if the identity operator $\mathrm{Id} : X \to X$ is compact.

An operator $P \in B(X)$ is called a *projection operator* if $P^2 = P$.

b. Let $P \in B(X)$ a projection operator. Prove that $PX \subset X$ is a closed linear subspace of X.

c. Prove that if $P \in B(X)$ is a compact projection operator, then PX is finite dimensional.

5. Let X be a Banach space, $f \in X', f \neq 0$ and $\alpha \in \mathbb{F}$. The *hyperplane in X defined by f and α* is the set

$$H = H_{(f,\alpha)} := \{x \in X : f(x) = \alpha\}.$$

a. Prove that $H_{(f,\alpha)} \neq \emptyset$ and that $|\alpha| \leq \|f\| \cdot \inf_{h \in H_{(f,\alpha)}} \|h\|$.

The following fact is a consequence of the Hahn-Banach theorem: Let X be a non-zero reflexive space with dual space X'. For every $f \in X'$ there exists an $x \in X$ with $\|x\| = 1$ such that $\|f\| = f(x)$.

b. Suppose that X is reflexive. Prove that there exists $z \in H_{(f,\alpha)}$ such that

$$\inf_{h \in H_{(f,\alpha)}} \|h\| = \|z\|.$$

Recall the isometric isomorphism

$$\varphi : \ell^\infty \simeq (\ell^1)', \quad \varphi(x)(y) := \sum_{n=1}^{\infty} x_n y_n, \quad x \in \ell^\infty, y \in \ell^1.$$

Let $x = (x_k) \in \ell^\infty$ be such that $\|x\|_\infty \notin \{|x_k| : k \in \mathbb{N}\}$ and consider the hyperplane $H_{(\varphi(x), \|x\|_\infty)} \subset \ell^1$.

c. Prove that for all $y \in \ell^1$ we have $|\varphi(x)(y)| < \|x\|_\infty \|y\|_1$.

d. Prove that there does not exist $y \in H_{(\varphi(x), \|x\|_\infty)}$ such that

$$\inf_{h \in H_{(\varphi(x), \|x\|_\infty)}} \|h\| = \|y\|.$$

Note that this proves that ℓ^1 is not reflexive.

Hint for d.) : Argue by contradiction.

Preliminary point distribution

Question:	1	2	3	4	5	Total
Points:	10 (2+3+2+3)	8 (2+2+4)	5 (2+3)	10 (4+2+4)	12 (2+4+2+4)	45

$$\text{Grade} := 1 + \frac{\text{(total number of points)}}{5}$$