Linear Analysis

6 January 2023 13:15-16:15hr

- You can use the results of the earlier parts of a question, even if you have not solved these parts.
- Write in full sentences and clearly state what you use in your solutions.
- The point distribution is preliminary and may be subject to change.
- This exam has five questions on two pages.

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1. Let X be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset X$ a sequence such that for all $f \in X'$ the series $\sum_{k=1}^{\infty} |f(x_k)|$ is convergent.

   a. The maps $T_n : X' \to \ell^1$ defined by

   $$T_n(f) := (f(x_1), f(x_2), \cdots, f(x_n), 0, 0, \cdots),$$

   are linear (you need not prove this). Show that each T_n is a bounded map.

   b. Prove that for all $f \in X'$ the sequence $T_n(f)$ is convergent.

   c. Prove that $\sup_n \|T_n\| < \infty$.

   It follows from b.) that $T : X' \to \ell^1$, $T(f) := \lim_{n\to\infty} T_n(f)$, is a well-defined linear map.

   d. Prove that $T : X' \to \ell^1$ is a bounded linear operator and that

   $$\sup_{f\in X', \|f\|\leq 1} \sum_{k=1}^{\infty} |f(x_k)| < \infty.$$

2. Let H be a Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$. Let $R, S : H \to H$ be two functions such that for all $x, y \in H$ we have $\langle x, Sy \rangle = \langle Rx, y \rangle$.

   a. Prove that R, S are linear maps.

   b. Show that for fixed $x \in H$, $\sup_{\|y\|=1} |\langle x, Sy \rangle| < \infty$.

   Recall the antilinear isometric isomorphism

   $$T_H : H \to H', \quad T_H(y)(x) := \langle x, y \rangle,$$

   so that $\|y\|_H = \|T_H(y)\|_{H'}$.

   c. Use the Uniform Boundedness Principle to show that S is a bounded map. Conclude that $R = S^*$, so that R is bounded as well.

3. Consider the Banach space $C([0, 1])$ of continuous functions on the interval $[0, 1]$ equipped with the norm $\|f\|_\infty := \sup_{t\in[0,1]} |f(t)|$. Denote by

   $$C^1([0, 1]) := \left\{ f \in C([0, 1]) : f' := \frac{\mathrm{d}f}{\mathrm{d}t} \in C([0, 1]) \right\} \subset C([0, 1]),$$

   the subset of functions whose derivative exists and is again continuous.

a. Prove that $C^1([0,1])$ is a vector space and that

$$\frac{\mathrm{d}}{\mathrm{d}t} : C^1([0,1]) \to C([0,1]), \quad f \mapsto \frac{\mathrm{d}f}{\mathrm{d}t},$$

is a linear map. Conclude that $(C^1([0,1]), \|\cdot\|_\infty)$ is a normed space.

b. Prove that the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ is not continuous.

Recall the following result: Let $f_n \in C^1([0,1])$ be a sequence and $f, g \in C([0,1])$. If f_n converges to f uniformly and f_n' converges to g uniformly then $f \in C^1([0,1])$ and $f' = g$.

c. Prove that the graph of the linear operator $\frac{\mathrm{d}}{\mathrm{d}t} : C^1([0,1]) \to C([0,1])$ is closed.

d. Use b.) and c.) to prove that $(C^1([0,1]), \|\cdot\|_\infty)$ is not a Banach space.

4. Let X be a real normed space and $A \subset X$ a nonempty subset. Define the *closed convex hull of* A to be the subset

$$\overline{\mathrm{conv}}(A) := \bigcap \{B \subset X : A \subset B,\ B \text{ is closed and convex}\}.$$

a. Prove that $\overline{\mathrm{conv}}(A)$ is a closed convex subset of X with the property that if $C \subset X$ is closed and convex, and $A \subset C$, then $\overline{\mathrm{conv}}(A) \subset C$.

b. Prove that

$$\overline{\mathrm{conv}}(A) = \left\{ x \in X : \text{for all } f \in X', \quad f(x) \le \sup_{a \in A} f(a) \right\}.$$

5. Let

$$c_0 := \left\{ x = (x_k)_{k=0}^{\infty} = (x_0, x_1, \cdots) : x_k \in \mathbb{F},\ \lim_{k\to\infty} x_k = 0 \right\},$$

be the space of all sequences converging to zero, supplied with the usual norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty := \sup_{k\ge 0} |x_k|$. Then c_0 is a Banach space (you need not prove this). This question will provide a proof of the fact that c_0 is not reflexive. Define $f : c_0 \to \mathbb{F}$ by setting

$$f(x) := \sum_{k=0}^{\infty} \frac{x_k}{k!}, \quad x = (x_0, x_1, \cdots) \in c_0.$$

a. Show that f is a well-defined *bounded* linear functional.

b. Show that $\|f\| = e$.

c. Show that for all $x \in c_0$ such that $\|x\| = 1$ it holds that $|f(x)| < e$.

The following fact is a consequence of the Hahn-Banach theorem: Let X be a non-zero reflexive space with dual space X'. For every $f \in X'$ there exists an $x \in X$ with $\|x\| = 1$ such that $\|f\| = f(x)$.

d. Using the above fact, prove that c_0 is not reflexive.

Preliminary point distribution

Question:	1	2	3	4	5	Total
Points:	8 (2+2+2+2)	8 (2+2+4)	10 (2+3+3+2)	9 (4+5)	10 (2+3+3+2)	45

$$\text{Grade} := 1 + \frac{\text{(total number of points)}}{5}$$