Retake examination for the course on

Random Walks

Teacher: Evgeny Verbitskiy

Wednesday, January 31, 2019, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or lecture notes is not allowed.
- There are 7 problems. The total number of points is 100 (per question indicated in boldface). A score of ≥ 50 points is necessary to pass the exam part. The final grade is the weighted average (75%) exam and (25%) homework grades

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(1) Consider a simple random walk $(S_n)_{n=0,\ldots,N}$ on $\mathbb{Z}$.

- (a) **[5]** Give a complete definition of a stopping time $T : \Omega_N \to \{0, 1, \ldots, N\}$.
- (b) **[10]** Sketch the proof the impossibility of the profitable stopping strategy. More specifically, if $T : \Omega_N \to \{0, 1, \ldots, N\}$ is a stopping time, then

$$\mathbb{E}(S_T) = 0.$$

(2) Suppose $\{S_n\}_{n=0}^{\infty}$ is a simple random walk in $d = 1$.

(a) [**5pt**] Show that for all $a, c \in \mathbb{N}$,

$$\mathbb{P}(S_n = a - c, \sigma_a \leq n) = \mathbb{P}(S_n = a + c).$$

(a) [**10pt**] Using the result of (a), show that for all $a, n \in \mathbb{N}$,

$$\mathbb{P}(\sigma_a \leq n) = \mathbb{P}(S_n \notin [-a, a - 1]).$$

(3) Consider a simple random walk $(S_n)_{n=0}^{\infty}$ on $\mathbb{Z}^d$.

(a) **[5]** Define a notion of recurrence of a random walk on the lattice $\mathbb{Z}^d$.

(b) **[10]** Argue that

$$G(0; z) = 1 + F(0; z)G(0; z),$$

where $G(x; z)$ is the Green function and

$$F(x; z) = \sum_{n \geq 0} \mathbb{P}(\sigma_x = n)z^n.$$

Formulate a criterion for the recurrence in terms of $F(0; 1)$.

(4) (a) **[5]** Formulate the Thompson Principle.

(b) **[5]** Formulate the Dirichlet Principle.

(c) **[5]** Derive the Dirichlet Principle from the Thomson Principle.

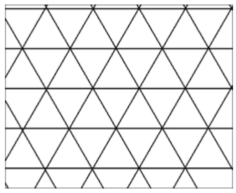
(5) (a) **[10]** Show that, for every point $x \in \mathbb{R}$, there exists a two-sided Brownian motion starting x, i.e., $\{W(t) : t \in \mathbb{R}\}$ with $W(0) = x$, which has continuous paths, independent increments and the property that, for all $t \in \mathbb{R}$ and $h > 0$, the increments $W(t+h) - W(t)$ are normally distributed with expectation zero and variance h.

(b) **[5]** Suppose $(W(t))_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}$. Is the process

$$\tilde{W}(0) = 0, \text{ and } \tilde{W}(t) = tW(1/t), t > 0,$$

again a standard Brownian motion?

(6) Denote by c_n the number of self-avoiding walks of length $n \in \mathbb{N}$ on the infinite triangular lattice:



(a) **[5pt]** What inequality is satisfied by c_n's, and why does this inequality imply the existence of the so-called connectivity constant μ?

(b) **[5pt]** Compute c_3.

(c) **[5pt]** Show that $2^n \leq c_n \leq 6 \times 5^{n-1}$ for all $n \in \mathbb{N}$, and use this to obtain bounds on μ.

(7) Suppose that the current price of a stock is $S_0 = 90$ euro, and that at the end of one period its price is either $S_1 = 60$ euro or $S_1 = 120$ euro. The interest rate is 10%.

(a) **[5]** Compute the arbitrage-free price of the European call option at strike price $K = 100$, expiring after one period.

(b) **[5]** Suppose you can purchase or sell the option for 15 euro. What should you do?