Linear Analysis

3 January 2019 10:00-13:00hr

- If you are not a native English speaker and you are in doubt about the meaning of the questions, please ask the invigilator.
- You can answer the questions in English or in Dutch. If your choice is Dutch, please feel free to use English terminology when convenient.
- You can use the results of the earlier parts of a question, even if you have not solved these parts.
- The point distribution is preliminary and may be subject to change.
- This exam has five questions on two pages.

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1. Let X be a normed space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous linear functionals on X such that $\sup_{n\geq 1} \|f_n\| < \infty$. Define the map $T : X \to \ell^{\infty}$ by setting

$$Tx \coloneqq (f_1(x), f_2(x), f_3(x), \dots)$$

for $x \in X$. Then T is obviously linear—this need not be shown.

3 pt. (a) Show that T is a *bounded* linear map.

4 pt. (b) Compute $\|T\|$.

Set

$$L \coloneqq \{\, x \in X : \lim_{n\to\infty} f_n(x) = 0 \,\}.$$

Then L is obviously a linear subspace of X—this need not be shown.

6 pt. (c) Show that L is a *closed* linear subspace of X. You may give a proof 'by hand' or use standard results about sequence spaces as you find convenient.

7 pt. 2. Let H be a separable complex Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and let f be a continuous linear functional on H.

Show that the series

$$\sum_{n=1}^{\infty} \overline{f(e_n)}\, e_n$$

converges in H. Hint: proving this will, at the same time, show what the sum of the series actually is.

3. Let X be a Banach space, and suppose that X has an at most countably infinite basis as a vector space. This question will show that X is actually finite dimensional, as follows. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of elements of X, with repetitions allowed, such that every element of X is a linear combination of finitely many of the x_n. For $k = 1, 2, \dots$, let L_k be the linear span of $x_1, \dots, x_k$.

3 pt. (a) Then $X = \bigcup_{k=1}^{\infty} L_k$. Why?

3 pt. (b) There exists at least one $k_0 \geq 1$ such that the closure of L_{k_0} has non-empty interior. Why?

5 pt. (c) If k_0 is as in part (b), then $X = L_{k_0}$. Why?

Exam continues overleaf

There exist infinite dimensional normed spaces that *do* have a countably infinite basis as a vector space.

4 pt. (d) Give an example of such a normed space.

4. Let X be a normed space, let L be a linear subspace of X, and suppose that $T_L : L \to \ell^\infty$ is a bounded linear operator from the subspace L into ℓ^∞. This question will show that T_L has a bounded linear extension to X, as follows.

4 pt. (a) Show that there exist bounded linear functionals $f_1, f_2, f_3, \ldots$ on L such that

$$T_L x = (f_1(x), f_2(x), f_3(x), \ldots)$$

for $x \in L$.

3 pt. (b) Show that $f_1, f_2, f_3, \ldots$ from part (a) are such that $\sup_{n\geq 1} \|f_n\| < \infty$.

4 pt. (c) Show that T_L has an extension to a bounded linear operator $T : X \to \ell^\infty$ that is defined on the whole space X. You may use part (a) of question 1 for this.

6 pt. 5. Let M be a metric space, and let S be a non-empty subset of M. Then $C_b(M, \mathbb{R})$, the space of all real-valued bounded continuous functions on M, is a Banach space when supplied with the norm

$$\|f\|_{\infty,M} := \sup_{x\in M} |f(x)|$$

for $f \in C_b(M, \mathbb{R})$. Likewise, $C_b(S, \mathbb{R})$, the space of all real-valued bounded continuous functions on S, is a Banach space when supplied with the norm

$$\|g\|_{\infty,S} := \sup_{x\in S} |g(x)|$$

for $g \in C_b(S, \mathbb{R})$. These facts need not be proved.
Let $R : C_b(M, \mathbb{R}) \to C_b(S, \mathbb{R})$ be the restriction map, defined by setting

$$(Rf)(x) := f(x)$$

for $f \in C_b(M, \mathbb{R})$ and $x \in S$. Suppose that R is surjective.
Exploit the fact that $C_b(M, \mathbb{R})$ and $C_b(S, \mathbb{R})$ are Banach spaces to show that there exists a constant $C > 0$ with the property that, for every $g \in C_b(S, \mathbb{R})$, there exists an $f \in C_b(M, \mathbb{R})$ such that:

(a) f extends g;

(b) $\|f\|_{\infty,M} \leq C\|g\|_{\infty,S}$.

Preliminary point distribution

Question:	1	2	3	4	5	Total
Points:	13 (3+4+6)	7 (7)	15 (3+3+5+4)	11 (4+3+4)	6 (6)	52

Grade := (total number of points)/5.2